M 325K

The Basis Step Worksheet

A proof by Mathematical Induction is used for proving a universal statement of the form

For all integers n $\geq$ an initial integer value, predicate P(n) is true.

For example, one such universal statement of this form is as follows:

$$\text{For all integers } n \geq 3, \quad 1^3 + 2^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2 .$$

The first step in writing a proof by Mathematical Induction is to write the Basis Step. In the Basis Step, it is proved that the predicate P(n) is a true statement when the value of the variable n is set to be equal to the initial integer value of n.

In this class, there are certain rules that the writing of the Basis Step of a proof by Mathematical Induction must follow. It is the purpose of this worksheet to teach you what the rules of the writing of the Basis Step are and how to write a Basis Step that follows these rules.

The Rules for Writing a Basis Step of a Proof by Mathematical Induction

In the Basis step of a proof by Mathematical Induction:

1) Using a "Let" statement, the first statement of the Basis Step must set the variable equal to its specified initial value.

2) Referring to the predicate P(n) of the universal statement-to-be-proved, every expression in the predicate that contains the variable must be evaluated individually, and each evaluation must proceed as follows:
First, the expression using the variable must be written as it appears in the predicate using the variable, and second, that expression is equated (by substitution) to the same expression with the initial integer value formally substituted for the variable, and third, the resulting calculation is simplified to an appropriate simple form.

3) After each expression containing the variable has been evaluated in this manner, the **assertion** of the predicate P(n) must be verified using the calculated values of the expressions found in part (2).

4) The final conclusion of the Basis Step is as follows:

"∴ For <variable> = <initial integer value>,
<predicate P(n) exactly as it appears in the statement-to-be-proved>, by substitution."

Note: You are not allowed to use the terminology "Property P(n)" as the Author Epp does in her proofs!

On the next pages are examples of Basis Step writing that follow the Rules for

Writing a Basis Step of a Proof by Mathematical Induction.

There is one example for each of three types of assertions of the predicate P(n):

(1) Assertion of Equality, (2) Assertion of Divisibility, (3) Assertion of Inequality

Examples of Basis Step Writing that follows the Rules for
Writing a Basis Step of a Proof by Mathematical Induction.

Example 1: (For a predicate P(n) that asserts an **Equality** Relationship)

To Prove: For all integers $n \geq 3, \quad 1^3 + 2^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2} \right]^2$.

Proof: [by Mathematical Induction]

[Basis Step] Let n = 3 .

$$1^3 + 2^3 + \cdots + n^3 = 1^3 + 2^3 + 3^3 \ \textit{by substitution,}$$
$$= 1 + 8 + 27$$
$$= 36 .$$

$$\left[\frac{n(n+1)}{2} \right]^2 = \left[\frac{3(3+1)}{2} \right]^2 \ \textit{by substitution,}$$

$$= (12/2)^2 = 6^2 = 36 .$$

36 = 36 .

$\therefore$ For $n = 3, \quad 1^3 + 2^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2} \right]^2$, by substitution. [End of Basis Step]

Example 2: (For a predicate P(n) that asserts a **Divisibility** Relationship)

To Prove: For all integers $n \geq 2, \quad 3^{2n} - 1$ *is divisible by* 8 .

Proof: [by Mathematical Induction]

[Basis Step] Let n = 2 .

$$3^{2n} - 1 = 3^{2\times 2} - 1 \ \textit{by substitution,}$$
$$= 3^4 - 1 = 81 - 1 = 80 .$$

$80 = 8 \times 10$.

$\therefore$ 80 is divisible by 8 .

$\therefore$ For $n = 2, \quad 3^{2n} - 1$ *is divisible by* 8, *by substitution.* [End of Basis Step]

Examples of Basis Step Writing that follows the Rules for
Writing a Basis Step of a Proof by Mathematical Induction. (continued)

Example 3: (For a predicate P(n) that asserts an **Inequality** Relationship)

To Prove: For all integers $n \geq 4, \quad 2^n < (n+2)!$.

Proof: [by Mathematical Induction]

[Basis Step] Let n = 4 .

$2^n = 2^4$ *by substitution,*

$= 16$.

$(n+2)! = (4+2)!$ *by substitution,*

$= 6!$

$= 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$

$16 < 720$.

$\therefore$ For $n = 4, \quad 2^n < (n+2)!$, *by substitution*. [End of Basis Step]

<u>The Basis Step Worksheet Problems:</u>

For each Universal Statement given, write the usual "To Prove:..../ Proof: ..." heading, and then write a Basis Step for a proof of the given universal statement by Mathematical Induction in such a manner that the writing of that Basis Step follows the Rules for Writing a Basis Step of a Proof by Mathematical Induction.

Problem (1): For all integers $n \geq 2, \quad 5^n + 9 < 6^n$.

Problem (2): For all integers $n \geq 2, \quad 2^{2n} - 1$ *is divisible by* 3 .

Problem (3): For all integers $n \geq 4, \quad \sum_{j=0}^{n} 2^j = 2^{(n+1)} - 1$